Math 1653W

8 November 2023

Warm-up: Give an equation for the line through the point $(3,5)$ with slope $\frac{1}{2}$.

Fast answer:

$$y = 5 + \frac{1}{2}(x-3)$$

(See Tasks 8–15 from List 0.)

Continuity

The function $f(x)$ is **continuous at $x = p$** if

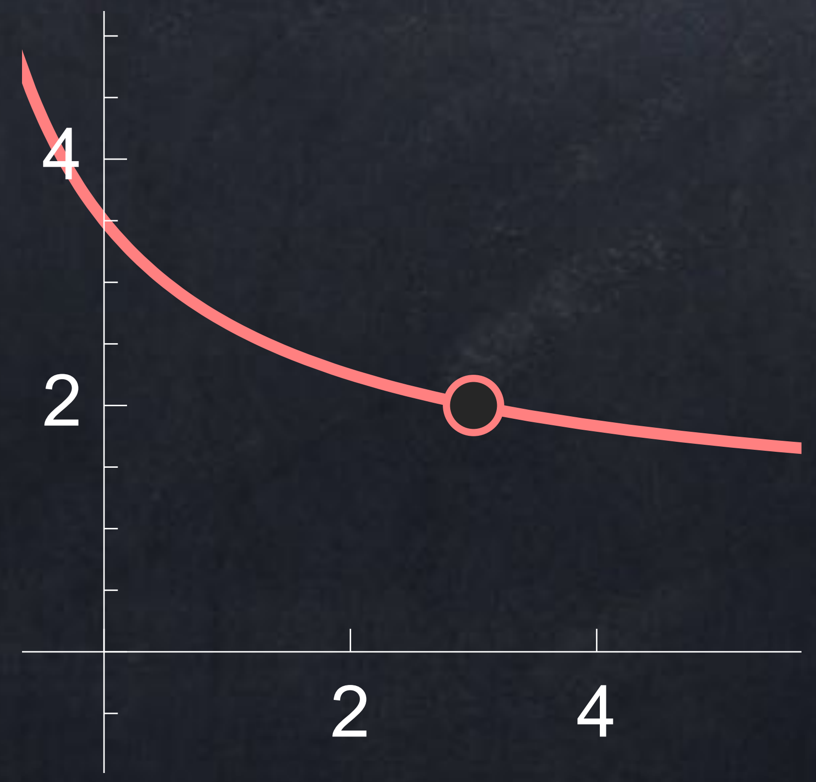
1. $f(p)$ is defined and 2. $\lim_{x \to p} f(x)$ exists and 3. $\lim_{x \to p} f(x) = f(p)$.

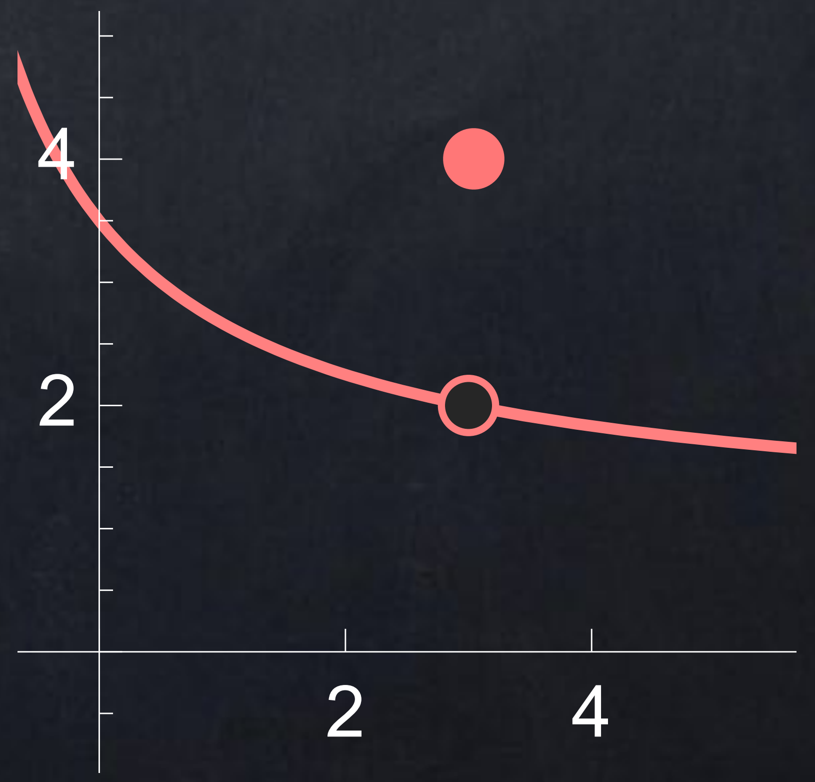
Many common functions (polynomials, exponentials, $\sin$ and $\cos$) are continuous at all points or on some interval ($\sqrt{x}$ is continuous on $[0, \infty)$).

Types of discontinuities

The graph $y = f(x)$ has a **hole** at $x = a$ if

1. $\lim_{x \to a} f(x)$ is finite and
2. $f(a)$ is not defined or $f(a) \neq \lim_{x \to a} f(x)$.

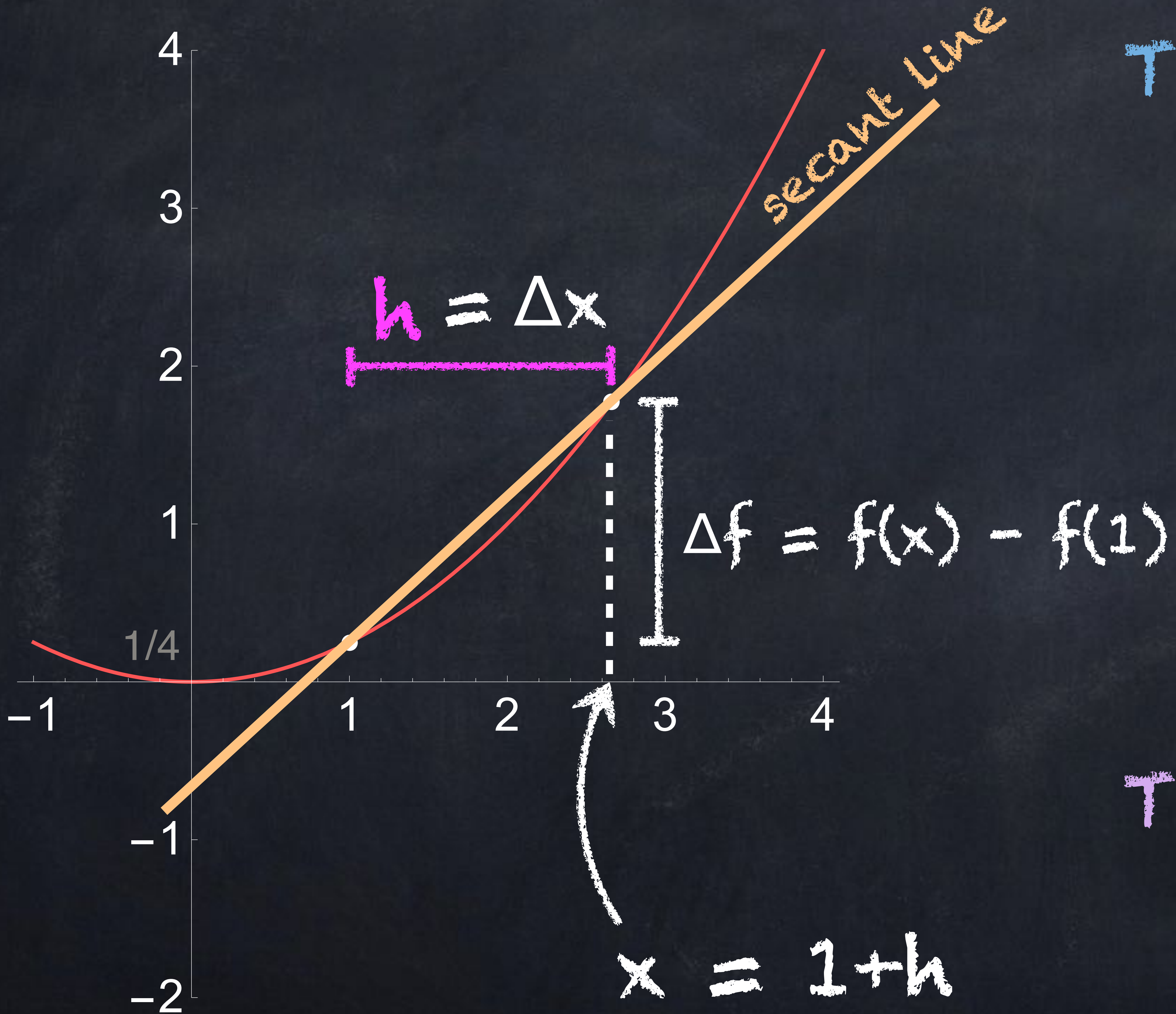




There are also official definitions for jump and asymptote.

Last time

What is the slope of the tangent line to $y = \frac{x^2}{4}$ at point $(1, \frac{1}{4})$?



The slope of the SECANT line is

$$\frac{\Delta f}{\Delta x} = \frac{f(x) - f(1)}{x - 1}$$

or

$$\frac{\Delta f}{\Delta x} = \frac{f(1+h) - f(1)}{h}$$

This is the same idea as

$$\text{avg. speed} = \frac{\Delta \text{position}}{\Delta \text{time}}$$

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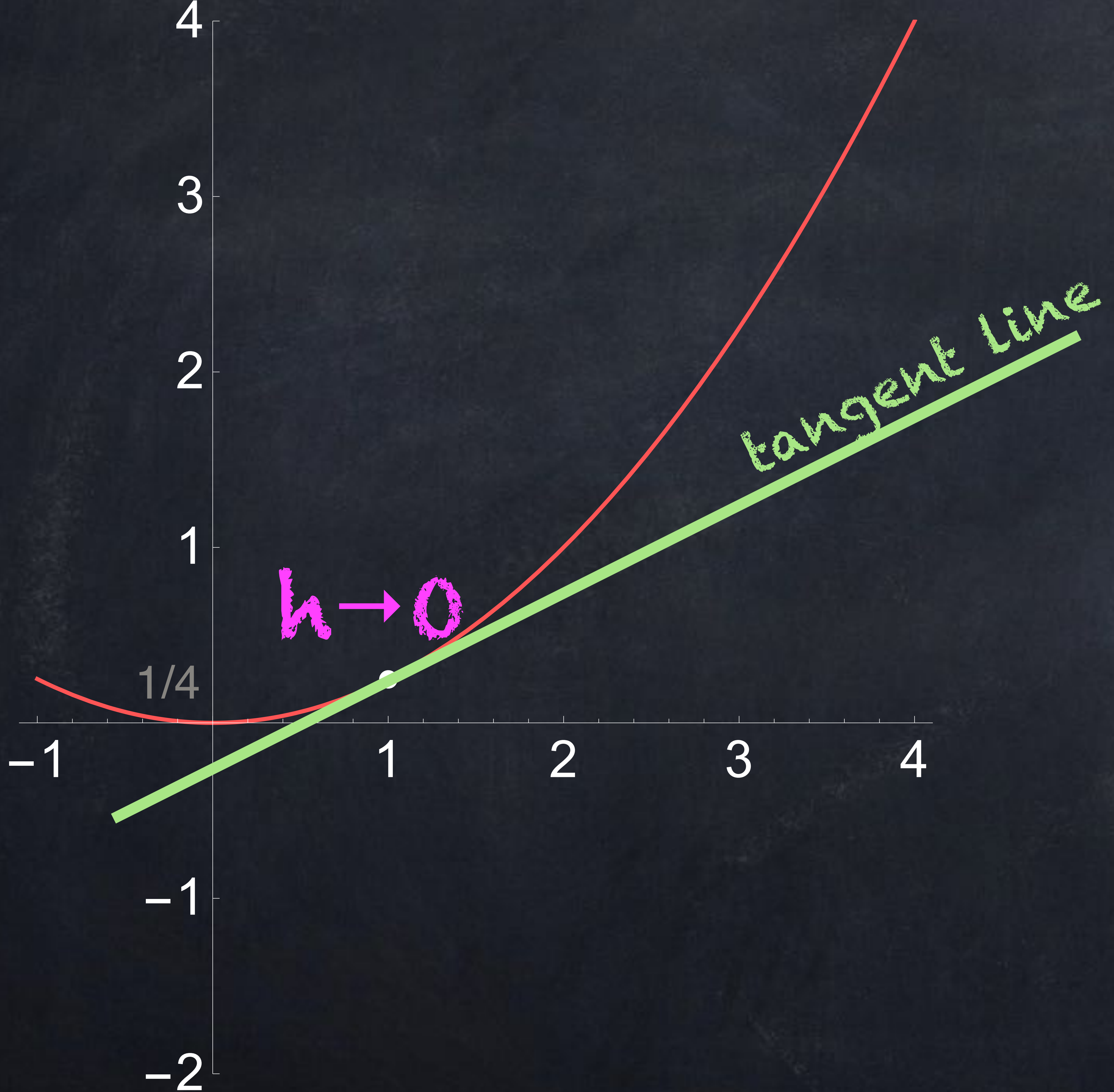


We can use either of these limits for the slope of the TANGENT line:

$$\text{slope} = \lim_{x \to 1} \frac{\frac{1}{4}(x)^2 - \frac{1}{4}}{x - 1}$$

$h = x - 1$

$x = h + 1$

$$\text{slope} \lim_{h \to 0} \frac{\frac{1}{4}(1+h)^2 - \frac{1}{4}}{h}$$

Derivative at a point

Last time

The **derivative of $f(x)$ at $x = a$** (or the **derivative of f at a**) can be written

- $f'(a)$ spoken as “F prime of A” or “F prime at A”
- $\left.\dfrac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=a}$ spoken as “D F D X at X equals A” or “D F D X when X=A”
- $\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=a}$ spoken as “D Y D X at X equals A” or “D Y D X when X=A”

and is calculated as $f'(a) = \lim_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$.

It is the slope of line through the point $(a, f(a))$ that is tangent to the graph of f.

Task: Give an equation for the tangent line to $y = \dfrac{x^2}{4}$ at $x = 1$.

- We have already calculated $f'(1) = \frac{1}{2}$ for the function $f(x) = x^2/4$.

$$\lim_{h \to 0} \frac{\frac{1}{4}(1+h)^2 - \frac{1}{4}}{h} = \frac{1}{2}$$

- We also know that $f(1) = \frac{1}{4}$, so the line includes the point $(1, \frac{1}{4})$.

The task is really give an equation for the line through $(1, \frac{1}{4})$ with slope $\frac{1}{2}$.

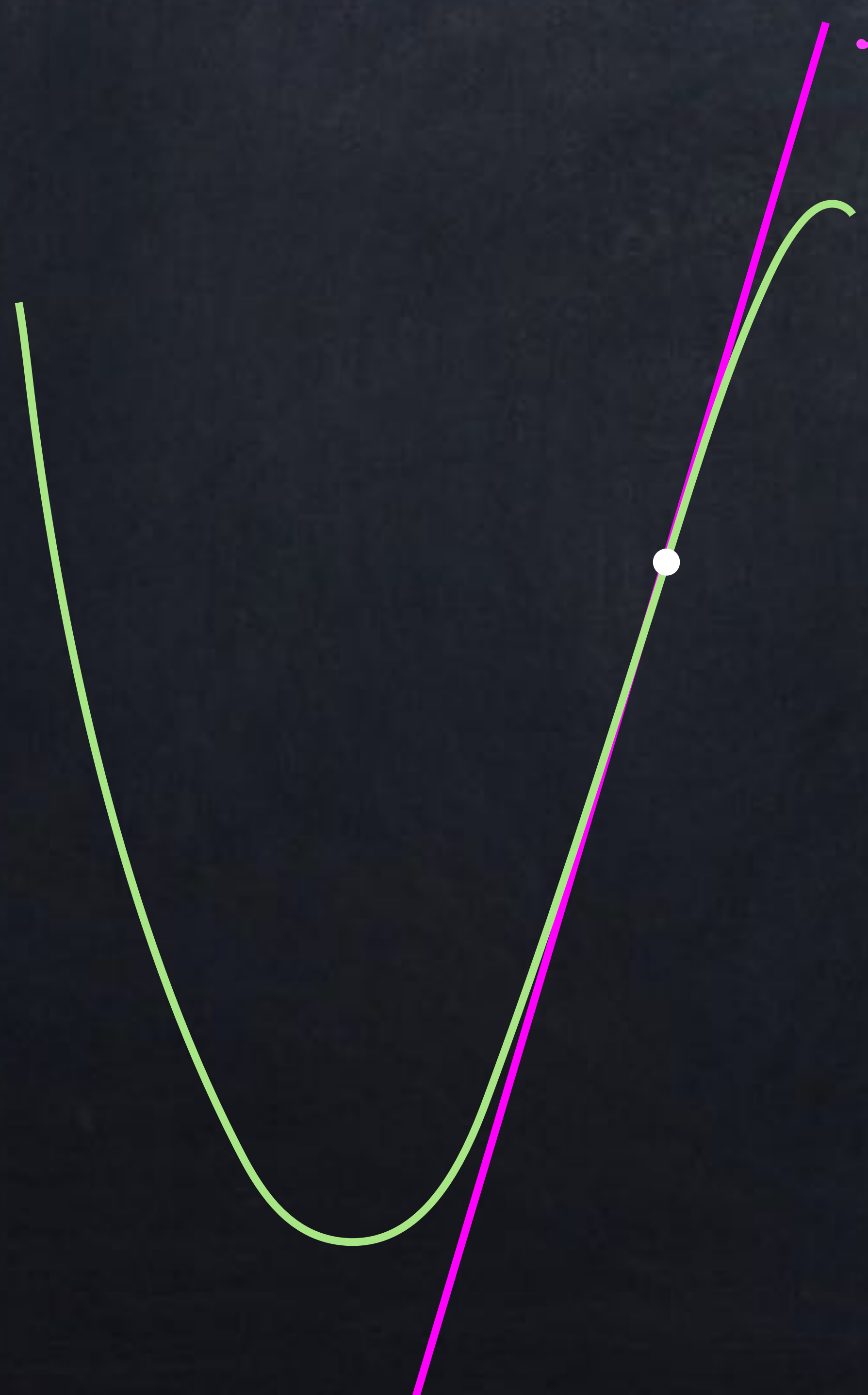
One answer is $y = \frac{1}{4} + \frac{1}{2}(x - 1)$

Linear approximation

Definition: the **linear approximation to $f(x)$ near $x = a$** is the function

$$L(x) = f(a) + f'(a)(x - a).$$

If you really understand tangent lines, then you already know this formula! It's what we need to make $y = L(x)$ tangent to $y = f(x)$.



- If x ≈ a (the value of x is close to a), then f(x) ≈ L(x).
- If x is far from a, then L(x) and f(x) might have very different values.

Tangent Lines

For $f(x) = \frac{-1}{10}x^5 + 6x^2 - \frac{4}{5}$, the derivative at $x = 2$ is

$$f'(2) = 16, \quad \text{or} \quad \left.\frac{df}{dx}\right|_{x=2} = 16.$$

Use this to..

- find the tangent line to $y = f(x)$ at $x = 2$.

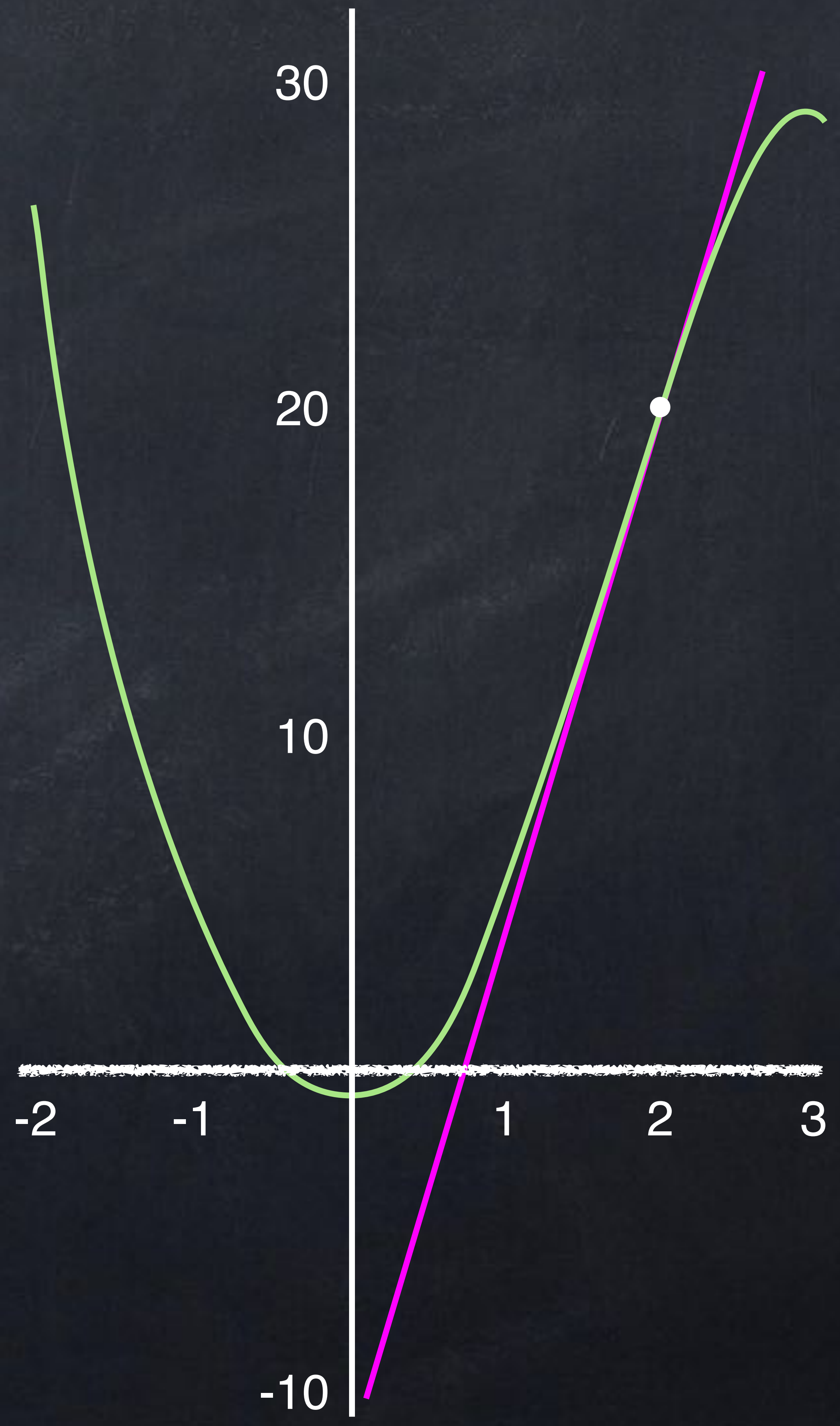
  First we need f(2) = -3.2 + 24 - 0.8 = 20

  Answer: y = 20 + 16(x - 2)

- approximate $f(2.25)$.

  Answer: 20 + 16(2.25 - 2) = 24

  (The actual value f(2.25) is 23.808...)

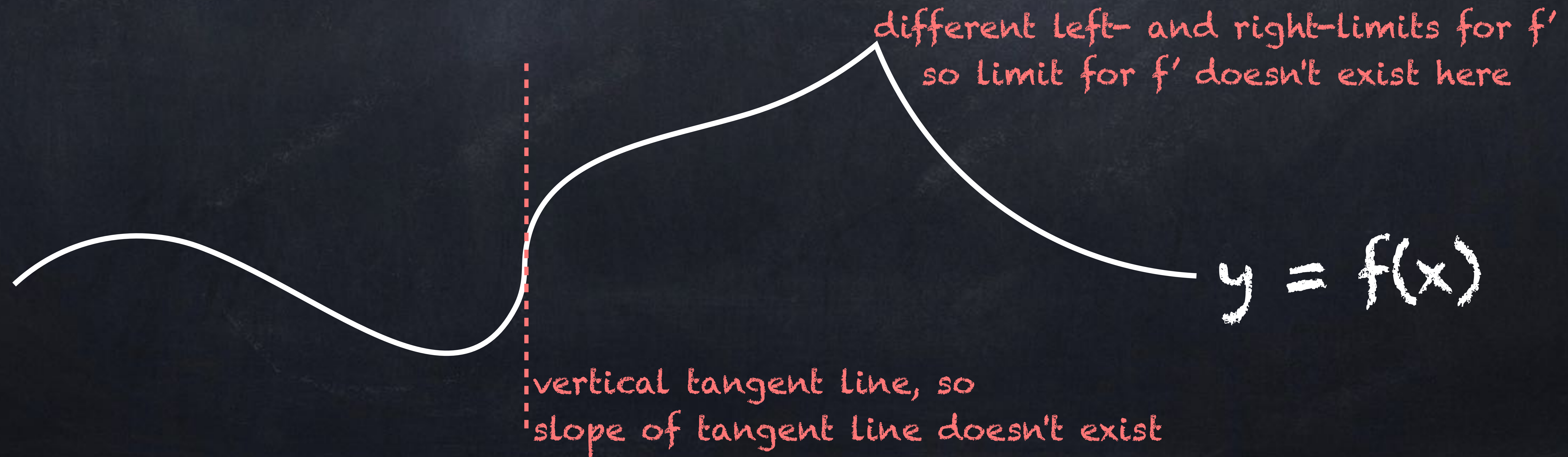


Continuous vs. differentiable

We know a function can be continuous at some points and not at others.

The same idea happens with derivatives.

- We say $f(x)$ is **differentiable at $x = a$** if $f'(a)$ is finite.
- We say $f(x)$ is **differentiable on $[a, b]$** if $f'(x)$ is finite for all $a \leq x \leq b$.



Continuous vs. differentiable

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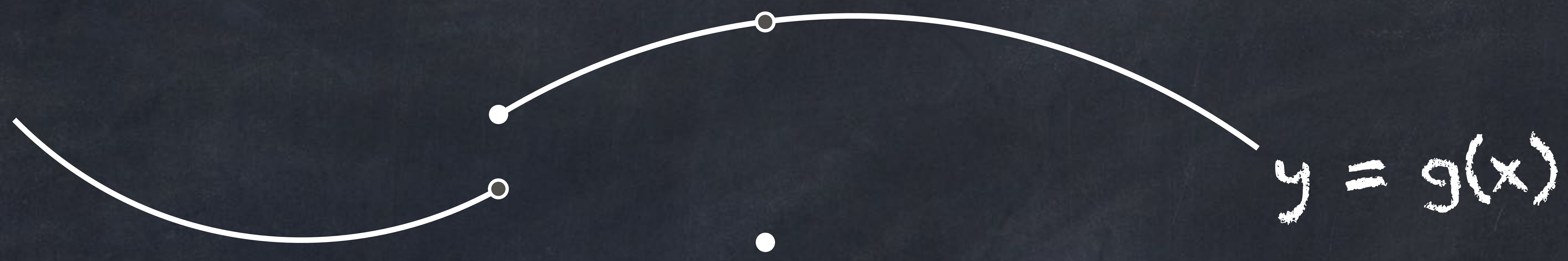


Fact: If a function is not continuous at $x = a$ then it is not differentiable at $x = a$.

Calculate $f'(14)$ for the function $f(x) = x^2$.

$$\lim_{h \to 0} \frac{(14+h)^2 - 14^2}{h} = \ldots = 2(14) = 28$$

- Calculate $f'(8)$ for the function $f(x) = x^2$.
- Calculate $f'(11)$ for the function $f(x) = x^2$.
- Calculate $f'(-3)$ for the function $f(x) = x^2$.

$\lim \ldots = 2(8) = 16$

$\lim \ldots = 2(11) = 22$

$\lim \ldots = 2(-3) = -6$

Derivatives as functions

The **derivative of $f(x)$ at $x = a$** (or the **derivative of f at a**) is a number equal to the slope of the tangent line to $y = f(x)$ at a.

The **derivative of $f(x)$** (or the **derivative of f** is) is a new function whose value at each point is the slope of the tangent line to f at that point.

There are several ways to write derivatives.

- $f'(x)$
- f'
- $D_x f$
- $\dfrac{df}{dx}$
- $\dfrac{dy}{dx}$

Formula: $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

Derivatives as functions

Finding the derivative of a function is called "differentiating the function".

- We differentiated x^2 and found that its derivative is $2x$.

  This can be written many ways:

  - $f'(x) = 2x$ for $f(x) = x^2$
  - $f' = 2x$ for $f(x) = x^2$
  - $\frac{\mathrm{d}f}{\mathrm{d}x} = 2x$ for $f(x) = x^2$
  - $\frac{\mathrm{d}}{\mathrm{d}x} f = 2x$ for $f(x) = x^2$
  - $D_x f = 2x$ for $f(x) = x^2$
  - $D_x \left(x^2\right) = 2x$
  - $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\right) = 2x$
  - $\left(x^2\right)' = 2x$

Derivative Rules

What is the derivative of $5x^2$?

$$\lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = 2x$$

$$\lim_{h\to 0} \frac{5(x+h)^2 - 5x^2}{h} = 5 \cdot 2x = 10x$$

because of how limits work with + and $\times$.

The Power Rule

The derivative of x^n is

$n\,x^{n-1}$

if n is any constant.

The Constant Multiple Rule

For any function f and constant c,

$$(c \cdot f(x))' = c \cdot f'(x).$$

The Sum Rule

For any functions f and g,

$$(f(x) + g(x))' = f'(x) + g'(x).$$

Example: Differentiate $3x^6 + 12\sqrt{x} + 4$.

Calculate the derivative of each of these, if you can:

- x^5 — $5x^4$
- x^{-3} — $-3x^{-4}$
- $\dfrac{7}{x}$ — $-7x^{-2}$
- 9
- $\sqrt{x}$ — $D[x^{1/2}] = \frac{1}{2}x^{-1/2}$
- 3^2 — $D[3^2] = D[9] = 0$
- 3^x — (We haven't learned this one yet.)
- $8x^3$ — $24x^2$

The Power Rule does NOT apply here since it's not in the form x^n.

Task 1: Calculate the derivative of $f(x) = 8x^3$ at $x = 2$.

$$f'(x) = 24x^2 \qquad f'(2) = \boxed{96}$$

Task 5: Calculate the derivative of $f(x) = \quad x \quad$ at $x = 0$.

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|h|}{h} \quad \boxed{\text{does not exist!}}$$

In the future we will learn how to differentiate ANY function. For example, the derivative of $x^3 \ln(\cos(5x^4))$ is $3x^2 \ln(\cos(5x^4)) - 20x^6 \tan(5x^4)$.

For now, we will only differentiate sums of powers. You <u>should</u> be able to calculate the derivative of

$$2x^{8/3} - 4x^5 + 27x + 12 + \frac{14}{x} + \sqrt{5x}$$

Where (if anywhere) is $y = \sqrt[3]{x}$ not continuous?

Where (if anywhere) is $y = \sqrt[3]{x}$ not differentiable?

Calculations vs. ideas

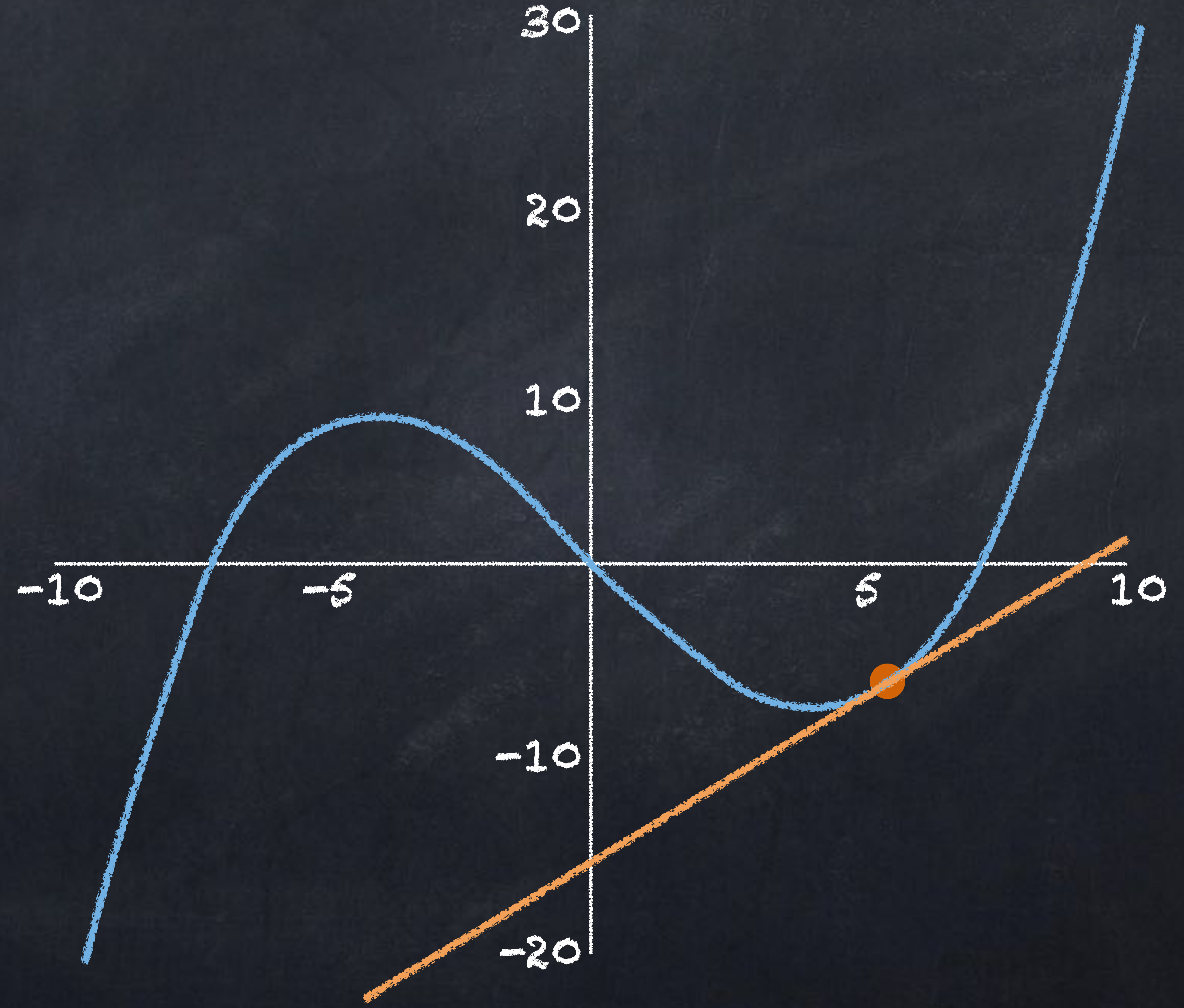
Given an equation for the tangent line to $y = \frac{1}{16}x^3 - 3x$ at the point $(6, \frac{-9}{2})$.

$$f(x) = \frac{1}{16}x^3 - 3x$$

$$f'(x) = \frac{3}{16}x^2 - 3$$

$$f'(6) = \frac{3 \times 36}{16} - 3 = \frac{15}{4}$$

$$y = \frac{-9}{2} + \frac{15}{4}(x-6)$$



Calculations vs. ideas

Find the "local maximum" of $\frac{1}{16}x^3 - 3x$.

$$f(x) = \frac{1}{16}x^3 - 3x$$

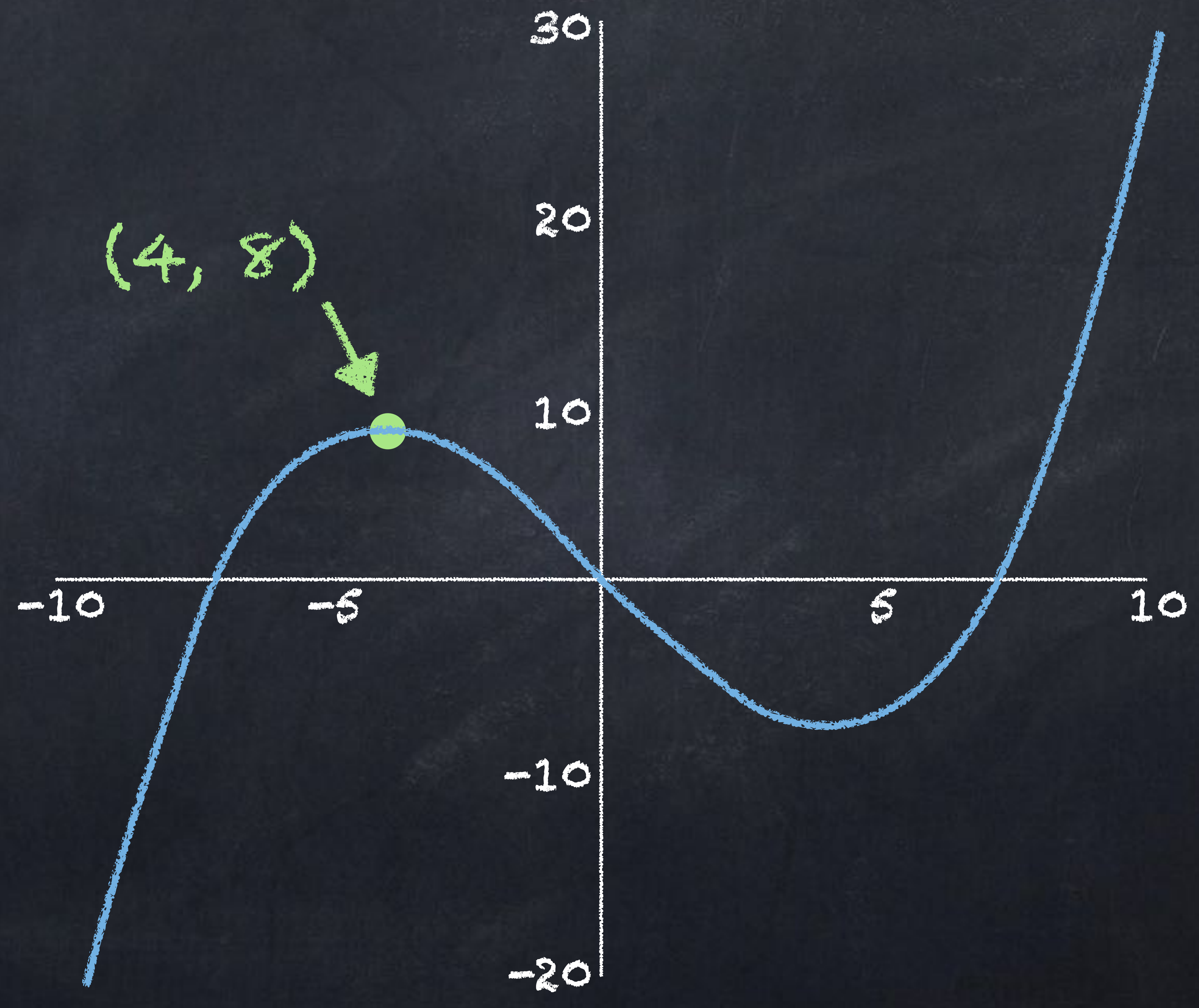
$$f'(x) = \frac{3}{16}x^2 - 3$$

...?...?...?...

(we will learn this later)

$$x = -4$$

$$f(-4) = \frac{-64}{16} + 12 = 8$$

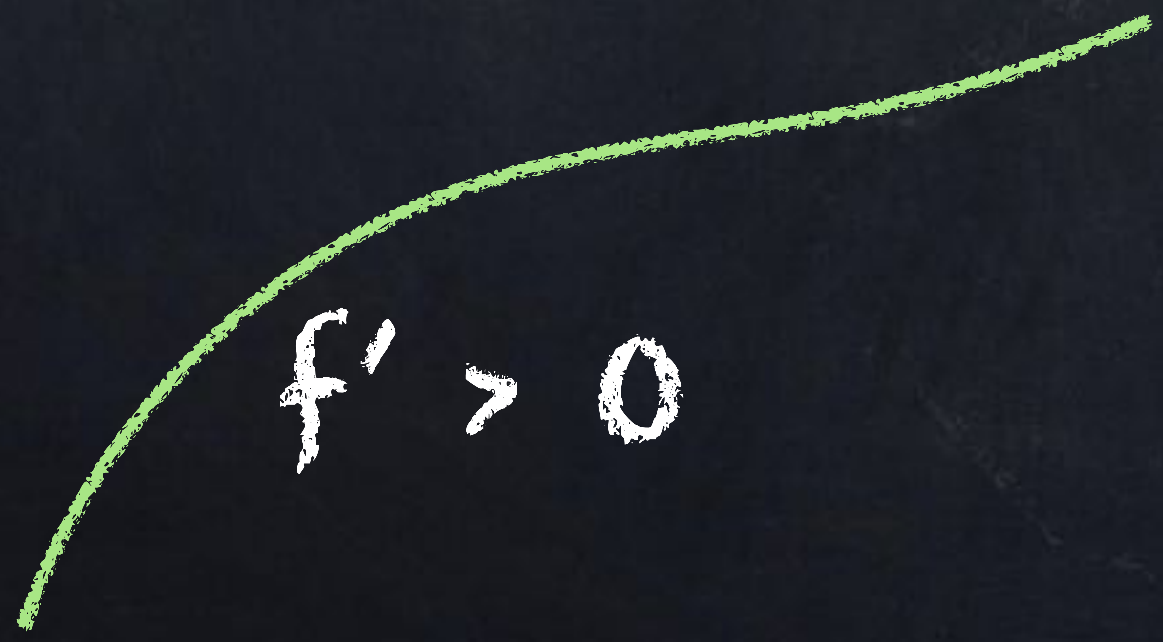


Increasing and decreasing

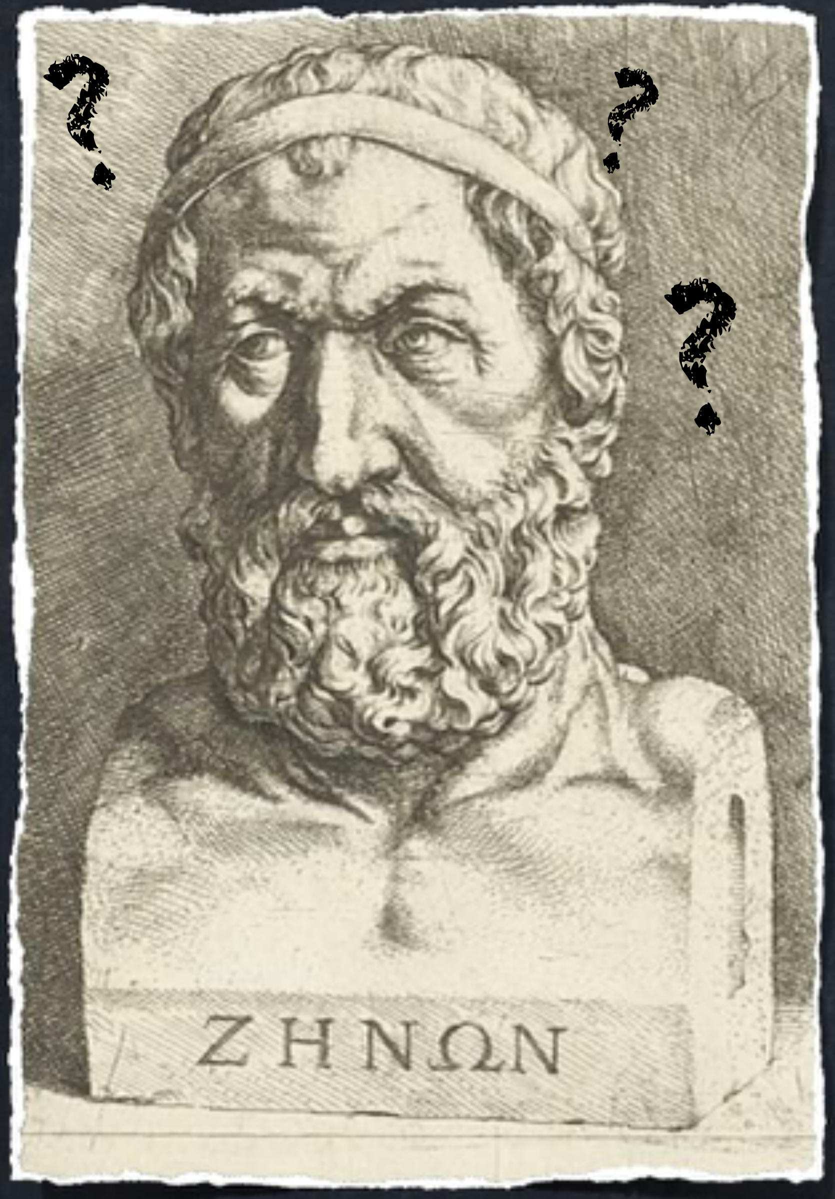
Definitions: We say $f(x)$ is <u>strictly **increasing** on an interval</u> if for any a, b in that interval with $a < b$ we have $f(b) > f(a)$.

We say $f(x)$ is strictly **decreasing** on an interval if … $f(b) < f(a)$.

Facts: If $f(x)$ is strictly increasing on an interval, then $f'(x) > 0$ for all x-values in that interval. If $f(x)$ is strictly decreasing, then $f'(x) < 0$.



Increasing and decreasing



Zeno of Elea wrote three Paradoxes of Motion, such as asking how an arrow can possibly move if it is motionless at every instant in time.

Today, limits and derivatives can resolve many of his questions.

Definition: We say $f(x)$ is **increasing at $x = c$** if $f'(c) > 0$, and we say $f(x)$ is **decreasing at $x = c$** if $f'(c) < 0$.

- You can argue philosophically about whether f can really be increasing when x is exactly c, but it is a helpful word to use.

Increasing and decreasing

- Intervals where this functions is increasing:

  $-1 < x < 2$
  and $4 < x < 5$

- Intervals where this function is decreasing:

  $x < -1$
  and $2 < x < 4$
  and $x > 5$

You can write $(-1,2) \cup (4,5)$ and $(-\infty,-1) \cup (2,4) \cup (5,\infty)$ if you prefer.

Critical points

A **critical point** of f is an x-value where $f'(x)$ is either zero or doesn't exist.

- zero → horizontal tangent line
- does't exist → vertical tangent line, or corner, or discontinuity

A function can only change from increasing to decreasing (or dec. to inc.) at a critical point.